# More on Inference for Two-Sample Data

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# Outline

# Two-sample inference

- Comparing the means of two distinct populations with respect to the same measurement.
- Examples:
  - SAT scores of high school A vs. high school B.
  - Severity of a disease in women vs. in men.
  - Heights of New Zealanders vs. heights of Ethiopians.
  - Coefficients of friction after wear of sandpaper A vs. sandpaper B.
- Notation:

| Sample | 1 | 2 |
|---|---|---|
| Sample size | $n_1$ | $n_2$ |
| True mean | $\mu_1$ | $\mu_2$ |
| Sample mean | $\overline{x}_1$ | $\overline{x}_2$ |
| True variance | $\sigma_1^2$ | $\sigma_2^2$ |
| Sample variance | $s_1^2$ | $s_2^2$ |

# $n_1 \geq 25$ and $n_2 \geq 25$, variances known

- We want to test $H_0 : \mu_1 - \mu_2 = \#$ with some alternative hypothesis
- If $\sigma_1^2$ and $\sigma_2^2$ are known, use the test statistic:

$$K = \frac{(\overline{x}_1 - \overline{x}_2) - \#}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}}$$

  which has a $N(0, 1)$ distribution if:
  - $H_0$ is true.
  - The sample 1 points are iid with mean $\mu_1$ and variance $\sigma_1^2$, the sample 2 points are iid with mean $\mu_2$ and variance $\sigma_2^2$, and the two samples are independent.
- The confidence intervals (2-sided, 1-sided upper, and 1-sided lower, respectively) for $\mu_1 - \mu_2$ are:

$$\left( (\overline{x_1} - \overline{x}_2) - z_{1-\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}},\ (\overline{x_1} - \overline{x}_2) + z_{1-\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \right)$$

$$\left( -\infty,\ (\overline{x_1} - \overline{x}_2) + z_{1-\alpha}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \right)$$

$$\left( (\overline{x_1} - \overline{x}_2) - z_{1-\alpha}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}},\ \infty \right)$$

# $n_1 \geq 25$ and $n_2 \geq 25$, variances UNknown

- If $\sigma_1^2$ and $\sigma_2^2$ are UNknown, use the test statistic:

$$K = \frac{(\overline{x}_1 - \overline{x}_2) - \#}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

- and confidence intervals for $\mu_1 - \mu_2$:

$$\left( (\overline{x_1} - \overline{x}_2) - z_{1-\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}},\ (\overline{x_1} - \overline{x}_2) + z_{1-\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \right)$$

$$\left( -\infty,\ (\overline{x_1} - \overline{x}_2) + z_{1-\alpha}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \right)$$

$$\left( (\overline{x_1} - \overline{x}_2) - z_{1-\alpha}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}},\ \infty \right)$$

# Example: packing weights

- A company research effort involved finding a workable geometry for molded pieces of a solid.
- One comparison made was between the weight (in grams) of molded pieces of a particular geometry that could be poured into a standard container, and the weight of irregularly shaped pieces (obtained through crushing), that could be poured into the same container.
- $n_1 = 24$ crushed pieces and $n_2 = 24$ molded pieces were made and weighed.
- $\mu_1$ is the true mean packing weight of the crushed pieces, and $\mu_2$ is the true mean packing weight of the molded pieces.
- I want to formally test the claim that the crushed weights are greater than the molded weights.

# Example: packing weights

| Molded | | Crushed |
|---:|:---:|:---|
| 7.9 | 11 | |
| 4.5, 3.6, 1.2 | 12 | |
| 9.8, 8.9, 7.9, 7.1, 6.1, 5.7, 5.1 | 12 | |
| 2.3, 1.3, 0.0 | 13 | |
| 8.0, 7.0, 6.5, 6.3, 6.2 | 13 | |
| 2.2, 0.1 | 14 | |
| | 14 | |
| 2.1, 1.2, 0.2 | 15 | |
| | 15 | |
| | 16 | 1.8 |
| | 16 | 5.8, 9.6 |
| | 17 | 1.3, 2.0, 2.4, 3.3, 3.4, 3.7 |
| | 17 | 6.6, 9.8 |
| | 18 | 0.2, 0.9, 3.3, 3.8, 4.9 |
| | 18 | 5.5, 6.5, 7.1, 7.3, 9.1, 9.8 |
| | 19 | 0.0, 1.0 |
| | 19 | |

# Example: packing weights

1. $H_0 : \mu_1 - \mu_2 = 0$, $H_a : \mu_1 - \mu_2 > 0$.
2. $\alpha = 0.05$
3. The test statistic is:

$$K = \frac{(\overline{x}_1 - \overline{x}_2) - 0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

- $n_1$ and $n_2$ are each $< 25$, but each sample is normally distributed enough to flex the "$n \geq 25$" rule and allow $n_1 = n_2 = 24$.
- Hence, it is enough to assume:
  - The crushed weights are iid with mean $\mu_1$ and variance $\sigma_1^2$.
  - The molded weights are iid with mean $\mu_2$ and variance $\sigma_2^2$.
  - The crushed weights are independent of the molded weights.
- Under these assumptions, $K \sim N(0, 1)$ under the null hypothesis.

# Example: packing weights

4. The moment of truth:

$$K = \frac{(\overline{x}_1 - \overline{x}_2) - 0}{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \frac{179.55 - 132.97 - 0}{\sqrt{\frac{(8.34)^2}{24} + \frac{(9.31)^2}{24}}} = 18.3$$

$$\begin{aligned} \text{p-value} &= P(Z > K) = 1 - \Phi(K) = 1 - \Phi(18.3) \\ &= 4 \times 10^{-75} \end{aligned}$$

5. With a p-value of $4 \times 10^{-75} < \alpha$, we reject $H_0$ in favor of $H_a$.
6. There is overwhelming evidence that more crushed solid material by weight can be poured into the container than molded solid material.

# Example: packing weights

- The analogous lower 95% confidence interval for $\mu_1 - \mu_2$ is:

$$\left( (\overline{x_1} - \overline{x}_2) - z_{1-\alpha}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}},\ \infty \right)$$
$$= \left( (179.55 - 132.97) - z_{0.95}\sqrt{\frac{(8.34)^2}{24} + \frac{(9.31)^2}{24}},\ \infty \right)$$
$$= (46.58 - 1.64 \cdot 2.55,\ \infty)$$
$$= (42.40,\ \infty)$$

- We're 95% confident that the true mean packing weight of crushed solids is at least 42.40 g greater than that of the molded solids.

# Your turn: anchor bolts

- An experiment carried out to study various characteristics of anchor bolts resulted in 78 observations on shear strength (kip) of 3/8-in. diameter bolts and 88 observations on strength of 1/2-in. diameter bolts.

| Variable | N | Mean | Median | TrMean | StDev | SEMean |
|---|---|---|---|---|---|---|
| diam 3/8 | 78 | 4.250 | 4.230 | 4.238 | 1.300 | 0.147 |

| Variable | Min | Max | Q1 | Q3 |
|---|---|---|---|---|
| diam 3/8 | 1.634 | 7.327 | 3.389 | 5.075 |

| Variable | N | Mean | Median | TrMean | StDev | SEMean |
|---|---|---|---|---|---|---|
| diam 1/2 | 88 | 7.140 | 7.113 | 7.150 | 1.680 | 0.179 |

| Variable | Min | Max | Q1 | Q3 |
|---|---|---|---|---|
| diam 1/2 | 2.450 | 11.343 | 5.965 | 8.447 |

- Let Sample 1 be the 1/2 in diameter bolts and Sample 2 be the 3/8 in diameter bolts.
- Using a significance level of $\alpha = 0.01$, find out if the 1/2 in bolts are more than 2 kip stronger (in shear strength) than the 3/8 in bolts.
- Calculate and interpret the appropriate 99% confidence interval to support the analysis.

# Answers: anchor bolts

- $n_1 = 88, \; n_2 = 78.$
- $\overline{x}_1 = 7.14, \; \overline{x}_2 = 4.25$
- $s_1 = 1.68, \; s_2 = 1.3$

1. $H_0 : \mu_1 - \mu_2 = 2, \; H_a : \mu_1 - \mu_2 > 2$
2. $\alpha = 0.01$
3. The test statistic is:

$$K = \frac{(\overline{x}_1 - \overline{x}_2) - 2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

- Assume:
  - $H_0$ is true.
  - Sample 1 points are drawn from iid $(\mu_1, \sigma_1^2)$ distributions.
  - Sample 2 points are drawn from iid $(\mu_2, \sigma_2^2)$ distributions.
  - Samples 1 and 2 are independent.
- Then, $K \sim N(0, 1)$

# Answers: anchor bolts

4. The moment of truth:

$$K = \frac{(\overline{x}_1 - \overline{x}_2) - 2)}{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \frac{(7.14 - 4.25) - 2}{\sqrt{\frac{(1.68)^2}{88} + \frac{(1.3)^2}{78}}} = 3.84$$

$$\text{p-value} = P(Z > K) = 1 - P(Z \leq K) = 1 - P(Z \leq 3.84)$$
$$= 1 - \Phi(3.84) \approx 0$$

5. With a p-value $\approx 0 < \alpha = 0.01$, we reject $H_0$ in favor of $H_a$.
6. There is overwhelming evidence that the 1/2 in anchor bolts are more than 2 kip stronger in shear strength than the 3/8 in bolts.

# Answers: anchor bolts

- I use a lower confidence interval for $\mu_1 - \mu_2$:

$$\left( (\overline{x_1} - \overline{x}_2) - z_{1-\alpha}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}},\ \infty \right)$$
$$= \left( (7.14 - 4.25) - z_{0.99} \cdot \sqrt{\frac{1.68^2}{88} + \frac{1.3^2}{78}},\ \infty \right)$$
$$= (2.89 - 2.33 \cdot 0.232,\ \infty)$$
$$= (2.35,\ \infty)$$

- We're 99% confident that the true mean shear strength of the 1/2 in anchor bolts is at least 2.35 kip more than the true mean shear strength of the 3/8 in anchor bolts.

# Outline

# Small samples and $\sigma_1^2 \approx \sigma_2^2$

- If $\sigma_1^2 \approx \sigma_2^2$, then we can use the **pooled sample variance**,

$$s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}$$

- A test statistic to test $H_0 : \mu_1 - \mu_2 = \#$ against some alternative is:

$$K = \frac{\overline{x}_1 - \overline{x}_2 - \#}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

- $K \sim t_{n_1 + n_2 - 2}$ assuming:
  - $H_0$ is true.
  - The sample 1 points are iid $N(\mu_1, \sigma_1^2)$, the sample 2 points are iid $N(\mu_2, \sigma_2^2)$, and the sample 1 points are independent of the sample 2 points.

# Small samples and $\sigma_1^2 \approx \sigma_2^2$



- $1 - \alpha$ confidence intervals (2-sided, 1-sided upper, and 1-sided lower, respectively) for $\mu_1 - \mu_2$ under these assumptions are of the form:

$$\left( (\overline{x_1} - \overline{x}_2) - t_{\nu,\ 1-\alpha/2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}},\ (\overline{x_1} - \overline{x}_2) + t_{\nu,\ 1-\alpha/2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \right)$$

$$\left( -\infty,\ (\overline{x_1} - \overline{x}_2) + t_{\nu,\ 1-\alpha} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \right)$$

$$\left( (\overline{x_1} - \overline{x}_2) - t_{\nu,\ 1-\alpha} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}},\ \infty \right)$$

where $\nu = n_1 + n_2 - 2$.

# Example: springs

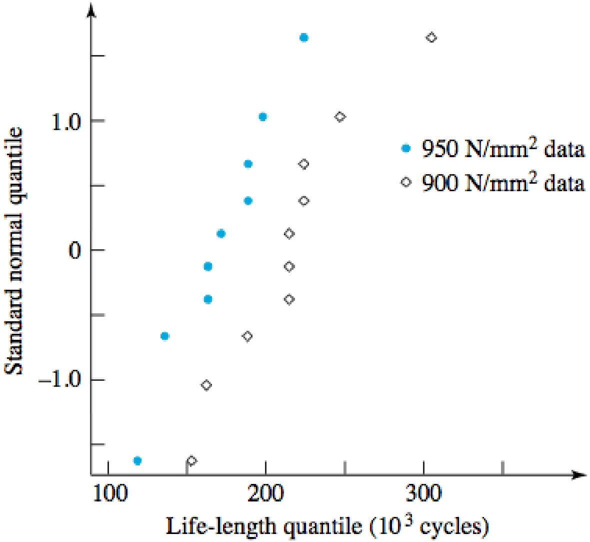
- The data of W. Armstrong on spring lifetimes (appearing in the book by Cox and Oakes) not only concern spring longevity at a 950 N/mm2 stress level but also longevity at a 900 N/mm2 stress level.

Spring Lifetimes under Two Different Levels of Stress ($10^3$ cycles)

| 950 N/mm$^2$ Stress | 900 N/mm$^2$ Stress |
|---|---|
| 225, 171, 198, 189, 189 | 216, 162, 153, 216, 225 |
| 135, 162, 135, 117, 162 | 216, 306, 225, 243, 189 |

- Let sample 1 be the 900 N/mm$^2$ stress group and sample 2 be the 950 N/mm$^2$ stress group.
- $\overline{x}_1 = 215.1, \overline{x}_2 = 168.3$.
- Let's do a hypothesis test to see if the sample 1 springs lasted significantly longer than the sample 2 springs.

First, since the samples are small, we need each sample to be roughly normally distributed.



# Example: springs

1. $H_0 : \mu_1 - \mu_2 = 0$, $H_a : \mu_1 - \mu_2 > 0$.
2. $\alpha = 0.05$
3. The test statistic is:

$$K = \frac{(\overline{x}_1 - \overline{x}_2) - 0}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

- Assume:
  - $H_0$ is true.
  - The sample 1 spring lifetimes are iid $N(\mu_1, \sigma_1^2)$
  - The sample 2 spring lifetimes are iid $N(\mu_2, \sigma_2^2)$
  - The sample 1 spring lifetimes are independent of those of sample 2.
- Under these assumptions, $K \sim t_{n_1+n_2-2} = t_{10+10-2} = t_{18}$.
- Reject $H_0$ if $K > t_{18,\ 1-\alpha}$

# Example: springs



$$
\begin{aligned}
s_1 &= \sqrt{\frac{1}{n_1 - 1} \sum_i (x_{1,i} - \overline{x}_1)^2} \\
&= \sqrt{\frac{1}{9}(225 - 215.1)^2 + (171 - 215.1)^2 + \cdots + (162 - 215.1)^2} = 42.9 \\
s_2 &= \sqrt{\frac{1}{n_2 - 1} \sum_i (x_{2,i} - \overline{x}_2)^2} \\
&= \sqrt{\frac{1}{9}(225 - 168.3)^2 + (171 - 168.3)^2 + \cdots + (162 - 168.3)^2} = 33.1 \\
s_p &= \sqrt{\frac{(10-1)42.9^2 + (10-1)33.1^2}{10 + 10 - 2}} = 38.3
\end{aligned}
$$

# Example: springs

4. The moment of truth:

$$K = \frac{(\overline{x}_1 - \overline{x}_2) - 0}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{215.1 - 168.3 - 0}{38.3 \cdot \sqrt{\frac{1}{10} + \frac{1}{10}}} = 2.7$$

$$\begin{aligned} t_{18,\ 1-\alpha} &= t_{18,\ 1-0.05} = t_{18,\ 0.95} \\ &= 1.73 \end{aligned}$$

5. With $K = 2.7 > 1.73 = t_{18,0.95}$, we reject $H_0$ in favor of $H_a$.
6. There is enough evidence to conclude that springs last longer if subjected to 900 $N/mm^2$ of stress than if subjected to 950 $N/mm^2$ of stress.

# Example: springs

- A 95%, 2-sided confidence interval for the difference in lifetimes is:

$$\left( (\overline{x_1} - \overline{x}_2) - t_{\nu,\ 1-\alpha/2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}},\ (\overline{x_1} - \overline{x}_2) + t_{\nu,\ 1-\alpha/2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \right)$$

Using $t_{\nu,\ 1-\alpha/2} = t_{18,1-0.05/2} = t_{18,\ 0.975} = 2.1$:

$$\left( (215.1 - 168.3) - 2.1 \cdot 38.3 \sqrt{\frac{1}{10} + \frac{1}{10}}, (215.1 - 168.3) + 2.1 \cdot 38.3 \sqrt{\frac{1}{10} + \frac{1}{10}} \right)$$
$$= (10.8, 82.8)$$

- We are 95% confident that the springs subjected to 900 $N/mm^2$ of stress last between $10.8 \times 10^3$ and $82.8 \times 10^3$ cycles longer than the springs subjected to 950 $N/mm^2$ of stress.

# Your turn: stopping distances

- Suppose $\mu_1$ and $\mu_2$ are true mean stopping distances (in meters) at 50 mph for cars of a certain type equipped with two different types of breaking systems.
- Suppose $n_1 = n_2 = 6, \overline{x}_1 = 115.7, \overline{x}_2 = 129.3, s_1 = 5.08, s_2 = 5.38$.
- Use significance level 0.01 to test $H_0 : \mu_1 - \mu_2 = -10$ vs. $H_a : \mu_1 - \mu_2 < -10$.
- Construct a 2-sided 99% confidence interval for the true difference in stopping distances.

# Answers: stopping distances

1. $H_0 : \mu_1 - \mu_2 = 0$, $H_a : \mu_1 - \mu_2 < -10$.
2. $\alpha = 0.01$
3. The test statistic is:

$$K = \frac{(\overline{x}_1 - \overline{x}_2) - (-10)}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

- Assume:
  - $H_0$ is true.
  - The sample 1 stopping distances are iid $N(\mu_1, \sigma_1^2)$
  - The sample 2 stopping distances are iid $N(\mu_2, \sigma_2^2)$
  - The sample 1 stopping distances are independent of those of sample 2.
- Under these assumptions, $K \sim t_{n_1+n_2-2} = t_{6+6-2} = t_{10}$.
- Reject $H_0$ if $K < t_{10,\ \alpha}$

# Answers: stopping distances

- $s_1 = 5.08$, $s_2 = 5.38$.
- $$s_p = \sqrt{\frac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}} = \sqrt{\frac{(6-1)(5.08)^2 + (6-1)(5.38)^2}{6+6-2}} = 5.23$$

# Answers: stopping distances

4. The moment of truth:

$$K = \frac{(\overline{x}_1 - \overline{x}_2) - (-10)}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{115.7 - 129.3 + 10}{5.23 \cdot \sqrt{\frac{1}{6} + \frac{1}{6}}} = -1.19$$

$t_{10,\ 1-\alpha} = t_{10,\ 0.99} = -2.76$

5. With $K = -1.19 \not< -2.76 = t_{10,0.99}$, we reject $H_0$ in favor of $H_a$.
6. There is not enough evidence to conclude that the stopping distances of breaking system 1 are less than those of breaking system 2 by over 10 m.

# Answers: stopping distances

- A 99%, 2-sided confidence interval for the difference in breaking distances is:

$$\left( (\overline{x_1} - \overline{x}_2) - t_{\nu,\ 1-\alpha/2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}},\ (\overline{x_1} - \overline{x}_2) + t_{\nu,\ 1-\alpha/2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \right)$$

Using $t_{\nu,\ 1-\alpha/2} = t_{10, 1-0.01/2} = t_{10,\ 0.995} = 3.17$:

$$\left( (115.7 - 129.3) - 3.17 \cdot 5.23 \sqrt{\frac{1}{6} + \frac{1}{6}}, (115.7 - 129.3) + 3.17 \cdot 5.23 \sqrt{\frac{1}{6} + \frac{1}{6}} \right)$$
$$= (-23.17, -4.03)$$

- We are 99% confident that the true mean stopping distance of braking system 1 is anywhere from 23.17 m to 4.03 m less than that of breaking system 2.

# What if $\sigma_1^2 \neq \sigma_2^2$?

- If $\sigma_1^2 \neq \sigma_2^2$, the distribution of the test statistic has an *approximate* t distribution with degrees of freedom estimated by the following special case of the Cochran-Satterthwaite approximation for linear combinations of mean squares:

$$\widehat{\nu} = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{s_1^4}{(n_1-1)n_1^2} + \frac{s_2^4}{(n_2-1)n_2^2}}$$

- The test statistic for testing $H_0 : \mu_1 - \mu_2 = \#$ vs. some $H_a$ is:

$$K = \frac{\overline{x}_1 - \overline{x}_2 - \#}{\sqrt{\frac{s_2^2}{n_2} + \frac{s_1^2}{n_1}}}$$

  which has a $t_{\widehat{\nu}}$ distribution under the assumptions that:
  - $H_0$ is true.
  - The sample 1 observations are iid $N(\mu_1, \sigma_1^2)$ and the sample 2 observations are iid $N(\mu_2, \sigma_2^2)$

# What if $\sigma_1^2 \neq \sigma_2^2$?

- Under these assumptions, the $1-\alpha$ confidence intervals for $\mu_1 - \mu_2$ become:

$$\left( (\overline{x_1} - \overline{x}_2) - t_{\widehat{\nu},\ 1-\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}},\ (\overline{x_1} - \overline{x}_2) + t_{\widehat{\nu},\ 1-\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \right)$$

$$\left( -\infty,\ (\overline{x_1} - \overline{x}_2) + t_{\widehat{\nu},\ 1-\alpha}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \right)$$

$$\left( (\overline{x_1} - \overline{x}_2) - t_{\widehat{\nu},\ 1-\alpha}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}},\ \infty \right)$$

# Example: springs



- In the springs example, $\sigma_1^2$ probably doesn't equal $\sigma_2^2$ because $s_1 = 57.9$ and $s_2 = 33.1$.
- I'll redo the hypothesis test and the confidence interval using:

$$\widehat{\nu} = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{s_1^4}{(n_1-1)n_1^2} + \frac{s_2^4}{(n_2-1)n_2^2}} = \frac{\left(\frac{57.9^2}{10} + \frac{33.1^2}{10}\right)^2}{\frac{57.9^4}{(10-1)10^2} + \frac{33.1^4}{(10-1)10^2}} = 14.3$$

# Example: springs

1. $H_0 : \mu_1 - \mu_2 = 0$, $H_a : \mu_1 - \mu_2 > 0$.
2. $\alpha = 0.05$
3. The test statistic is:

$$K = \frac{(\overline{x}_1 - \overline{x}_2) - 0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

- Assume:
  - $H_0$ is true.
  - The sample 1 spring lifetimes are $N(\mu_1, \sigma_1^2)$
  - The sample 2 spring lifetimes are $N(\mu_2, \sigma_2^2)$
  - The sample 1 spring lifetimes are independent of those of sample 2.
- Under these assumptions, $K \sim t_{\widehat{\nu}} = t_{14.3}$.
- Reject $H_0$ if $K > t_{14.3,\ 1-\alpha}$

# Example: springs

4. The moment of truth:

$$K = \frac{(\overline{x}_1 - \overline{x}_2) - 0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{215.1 - 168.3 - 0}{\sqrt{\frac{57.9^2}{10} + \frac{33.1^2}{10}}} = 2.22$$

$$t_{14.3,\ 1-\alpha} = t_{14.3,\ 1-0.05} = t_{14.3,\ 0.95}$$

$$= 1.76 \quad \text{(Take } \nu = 14 \text{ if you're using the t table)}$$

5. With $K = 2.22 > 1.76 = t_{14.3,0.95}$, we reject $H_0$ in favor of $H_a$.
6. There is still enough evidence to conclude that springs last longer if subjected to 900 $N/mm^2$ of stress than if subjected to 950 $N/mm^2$ of stress.

# Example: springs

- A 95%, 2-sided confidence interval for the difference in lifetimes is:

$$\left( (\overline{x_1} - \overline{x}_2) - t_{\widehat{\nu},\ 1-\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}},\ (\overline{x_1} - \overline{x}_2) + t_{\widehat{\nu},\ 1-\alpha/2}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \right)$$

Using $t_{\widehat{\nu},\ 1-\alpha/2} = t_{14.3,\ 1-0.05/2} = t_{14.3,0.975} = 2.14$:

$$\left( (215.1 - 168.3) - 2.14 \cdot \sqrt{\frac{57.9^2}{10} + \frac{33.1^2}{10}}, \right.$$

$$\left. (215.1 - 168.3) + 2.14 \cdot \sqrt{\frac{57.9^2}{10} + \frac{33.1^2}{10}} \right)$$

$$= (1.67,\ 91.9)$$

- We are 95% confident that the springs subjected to 900 $N/mm^2$ of stress last between $1.67 \times 10^3$ and $91.1 \times 10^3$ cycles longer than the springs subjected to 950 $N/mm^2$ of stress.

# Your turn: fabrics



- The void volume within a textile fabric affects comfort, flammability, and insulation properties. Permeability ($cm^3/cm^2/s$) of a fabric refers to the accessibility of void space to the flow of a gas or liquid.
- Consider the following data on two different types of plain-weave fabric:

| Fabric Type | Sample Size | Sample Mean | Sample Standard Deviation |
|---|---|---|---|
| Cotton | 10 | 51.71 | .79 |
| Triacetate | 10 | 136.14 | 3.59 |

- Let Sample 1 be the triacetate fabric and Sample 2 be the cotton fabric.
- Using $\alpha = 0.05$, attempt to verify the claim that triacetate fabrics are more permeable than the cotton fabrics on average.
- Construct and interpret a two-sided 95% confidence interval for the true difference in mean permeability.

# Answers: fabrics

- $n_1 = n_2 = 10$.
- $\overline{x}_1 = 136.14$, $\overline{x}_2 = 51.71$.
- $s_1 = 3.59$, $s_2 = 0.79$.
- $$\widehat{\nu} = \frac{\left(\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}\right)^2}{\frac{s_1^4}{(n_1-1)n_1^2} + \frac{s_2^4}{(n_2-1)n_2^2}} = \frac{\left(\frac{3.59^2}{10} + \frac{0.79^2}{10}\right)^2}{\frac{3.59^4}{(10-1)10^2} + \frac{0.79^4}{(10-1)10^2}} = 9.87$$
- If you're using the t table, round down to $\nu = 9$ to avoid unneccessary false positives.

# Answers fabrics

1. $H_0 : \mu_1 - \mu_2 = 0$, $H_a : \mu_1 - \mu_2 > 0$.
2. $\alpha = 0.05$
3. The test statistic is:

$$K = \frac{(\overline{x}_1 - \overline{x}_2) - 0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

- Assume:
  - $H_0$ is true.
  - The triacetate permeabilities are $N(\mu_1, \sigma_1^2)$
  - The cotton permeabilities are $N(\mu_2, \sigma_2^2)$
  - The triacetate permeabilities are independent of the cotton permeabilities.
- Under these assumptions, $K \sim t_{\widehat{\nu}} = t_{9.87}$.
- Reject $H_0$ if $K > t_{9.87,\ 1-\alpha}$

# Answers fabrics

4. The moment of truth:

$$K = \frac{(\overline{x}_1 - \overline{x}_2) - 0}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}} = \frac{136.14 - 51.71 - 0}{\sqrt{\frac{3.59^2}{10} + \frac{0.79^2}{10}}} = 72.63$$

$$t_{9.87,\ 1-\alpha} \approx t_{9,1-\alpha} = t_{9,\ 0.95} = 1.83$$

5. With $K = 72.63 > 1.83 = t_{9,0.95}$, we reject $H_0$ in favor of $H_a$.
6. There is overwhelming evidence to conclude that the triacetate fabrics are more permeable than the cotton fabrics.

# Answers fabrics

- With $t_{\widehat{\nu},1-\alpha/2} \approx t_{9,0.975} = 2.26$, a 95%, 2-sided confidence interval for the difference in lifetimes is:

$$\left( (\overline{x_1} - \overline{x}_2) - t_{\widehat{\nu},\ 1-\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}},\ (\overline{x_1} - \overline{x}_2) + t_{\widehat{\nu},\ 1-\alpha/2}\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} \right)$$

$$\left( (136.14 - 51.71) - 2.26 \cdot \sqrt{\frac{3.59^2}{10} + \frac{0.79^2}{10}}, \right.$$

$$\left. (136.14 - 51.71) + 2.26 \cdot \sqrt{\frac{3.59^2}{10} + \frac{0.79^2}{10}} \right)$$

$$= (81.80,\ 87.06)$$

- We are 95% confident that the permeability of the triacetate fabric exceeds that of the cotton fabric by anywhere between 81.80 $cm^3/cm^2/s$ and 87.06 $cm^3/cm^3/s$.