# Continuous Random Variables: Quantiles, Expected Value, and Variance 

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## Outline

Continuous Random Variables: Quantiles,
Expected Value, and Variance

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## Quantiles

## Quantiles

Expected Value
Variance
Functions of
Expected Value
random variables

## Functions of random variables

## Quantiles of continuous distributions

- The $p$-quantile of a random variable, X , is the number, $Q(p)$, such that:

$$
P(X \leq Q(p))=p
$$

- In terms of the cumulative distribution function (cdf):

$$
\begin{aligned}
F(Q(p)) & =p \\
Q(p) & =F^{-1}(p)
\end{aligned}
$$

## Example

- Let $Y$ be the time delay (s) between a 60 Hz AC circuit and the movement of a motor on a different circuit.

$$
f(y)= \begin{cases}60 & 0<y<\frac{1}{60} \\ 0 & \text { otherwise }\end{cases}
$$

$$
\begin{aligned}
0.95 & =P(Y \leq Q(0.95))=\int_{-\infty}^{Q(0.95)} f(y) d y \\
& =\int_{-\infty}^{0} 0 d x+\int_{0}^{Q(0.95)} 60 d y=0+\left(\left.60\right|_{0} ^{Q(0.95)}\right. \\
& =60 Q(0.95) \\
Q(0.95) & =\frac{0.95}{60}=\frac{19}{1200} \approx 0.0158
\end{aligned}
$$

Interpretation: on average, $95 \%$ of the time delays will be below 0.0158 seconds.

## Example

- You can also calculate quantiles directly from the cdf:

$$
F(y)= \begin{cases}0 & y \leq 0 \\ 60 y & 0<y \leq \frac{1}{60} \\ 1 & y>\frac{1}{60}\end{cases}
$$

- $Q(0.25):$

$$
\begin{aligned}
0.25 & =P(Y \leq Q(0.25))=F(Q(0.25)) \\
& =60 \cdot Q(0.25)
\end{aligned}
$$

Hence:

$$
Q(0.25)=\frac{0.25}{60}=\frac{1}{240} \approx 0.00417
$$

Interpretation: on average, $25 \%$ of the time delays will be below 0.00417 seconds.

## Your turn: calculating quantiles

- $T \sim \operatorname{Exp}(\alpha=1 / 2):$

$$
f(t)=\left\{\begin{array} { l l } 
{ 0 } & { t \leq 0 } \\
{ 2 e ^ { - 2 t } } & { t \geq 0 }
\end{array} \quad F ( t ) \left\{\begin{array}{ll}
0 & t<0 \\
1-e^{-2 t} & t \geq 0
\end{array}\right.\right.
$$

- Find:

1. $Q(0.05)$
2. $Q(0.5)$
3. $Q(p)$ for some $p$ with $0 \leq p \leq 1$

## Answers: calculating quantiles

1. $Q(0.05)$ :

$$
\begin{aligned}
0.05 & =P(T \leq Q(0.05))=F(Q(0.05))=1-e^{-2 Q(0.05)} \\
0.95 & =e^{-2 Q(0.05)} \\
\log (0.95) & =-2 Q(0.05) \\
Q(0.05) & =\frac{\log (0.95)}{-2} \approx 0.0256
\end{aligned}
$$

2. $Q(0.5)$ :

$$
\begin{aligned}
0.5 & =P(T \leq Q(0.5))=F(Q(0.5))=1-e^{-2 Q(0.5)} \\
0.5 & =e^{-2 Q(0.5)} \\
\log (0.5) & =-2 Q(0.5) \\
Q(0.5) & =\frac{\log (0.5)}{-2} \approx 0.347
\end{aligned}
$$

## Answers: calculating quantiles

3. $Q(p)$

$$
\begin{aligned}
p & =P(T \leq Q(p))=F(Q(p))=1-e^{-2 Q(p)} \\
1-p & =e^{-2 Q(p)} \\
\log (1-p) & =-2 Q(p) \\
Q(p) & =\frac{\log (1-p)}{-2}
\end{aligned}
$$

## Outline

## Functions of random variables

## Expected value

- The expected value of a continuous random variable is:

$$
E(X)=\int_{-\infty}^{\infty} x \cdot f(x) d x
$$

- As with continuous random variables, $E(X)$ (often denoted by $\mu$ ) is the mean of $X$, a measure of center.


## Example: time delay, $Y$

$$
\begin{aligned}
& f(y)= \begin{cases}60 & 0 \leq y \leq \frac{1}{60} \\
0 & \text { otherwise }\end{cases} \\
& E(Y)=\int_{-\infty}^{\infty} y \cdot f(y) d y \\
& =\int_{-\infty}^{0} y \cdot 0 d y+\int_{0}^{1 / 60} y \cdot 60 d y+\int_{1 / 60}^{\infty} y \cdot 0 d y \\
& =0+\left(\frac{y^{2}}{2} \cdot 60\right)_{0}^{1 / 60}+0 \\
& =
\end{aligned}
$$

$E(X)$ is the "center of mass" of a distribution


## Your turn: calculate $\mathrm{E}(X)$

$$
f(x)= \begin{cases}0 & x<0 \\ \frac{1}{\alpha} e^{-x / \alpha} & x \geq 0\end{cases}
$$

1. $X \sim \operatorname{Exp}(3)$
2. $X \sim \operatorname{Exp}(\alpha)$

## Answers: Calculate $\mathrm{E}(X)$

1. $X \sim \operatorname{Exp}(3):$

$$
\begin{aligned}
E(X) & =\int_{-\infty}^{\infty} x \cdot f(x) d x \\
& =\int_{-\infty}^{0} x \cdot 0 d x+\int_{0}^{\infty} x \cdot \frac{1}{3} e^{-x / 3} d x
\end{aligned}
$$

Quantiles
Expected Value

## Variance

Functions of
integration by parts:

$$
\begin{aligned}
& =0+\left(x\left(-e^{-x / 3}\right)\right)_{0}^{\infty}-\int_{0}^{\infty}\left(-e^{-x / 3}\right) d x \\
& =\left(-\infty e^{-\infty / 3}+0 e^{-0 / 3}\right)+\int_{0}^{\infty} e^{-x / 3} d x \\
& =0+\left(-3 e^{-x / 3}\right)_{0}^{\infty} \\
& =\left(-3 e^{-\infty / 3}+3 e^{-0 / 3}\right) \\
& =3
\end{aligned}
$$

2. Similarly, $\mathrm{E}(\mathrm{X})=\alpha$ when $X \sim \operatorname{Exp}(\alpha)$.

Example: waiting time for the next student to arrive at the library

- From 12:00 to 12:10 PM, about 12.5 students per minute enter on average.
- Hence, the average waiting time for the next student is $\frac{1}{12.5}=0.08$ minutes for the next student.
- Let $T \sim \operatorname{Exp}(0.08)$ be the time until the next student arrives.
- P (wait is more than 10 seconds $)=$

$$
P(T>1 / 6)=1-F(1 / 6)=1-\left(1-e^{(-0.08 \cdot 1 / 6)}\right)=0.12
$$

## Variance

Functions of
random variables


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## Variance

## Functions of random variables

## Variance

- The variance of a continuous random variable $X$ is:

$$
\operatorname{Var}(X)=\int_{-\infty}^{\infty}(x-E(X))^{2} \cdot f(x) d x
$$

Quantiles
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random variables

$$
\begin{aligned}
\operatorname{Var}(X) & =\int_{-\infty}^{\infty} x^{2} f(x) d x-E^{2}(X) \\
& =E\left(X^{2}\right)-E^{2}(X)
\end{aligned}
$$

- The standard deviation is $\operatorname{SD}(X)=\sqrt{\operatorname{Var}(X)}$


## Your turn: checkout time

Let $X$ denote the amount of time for which a book on 2-hour reserve at a college library is checked out by a randomly selected student and suppose that $X$ has density function

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$$
f(x)= \begin{cases}.5 x & 0 \leq x \leq 2 \\ 0 & \text { otherwise }\end{cases}
$$

Calculate:

1. $\mathrm{E}(X)$
2. $\operatorname{Var}(X)$

## Answers: checkout time

1. 

$$
\begin{aligned}
E(X) & =\int_{-\infty}^{\infty} x \cdot f(x) d x=\int_{0}^{2} x \cdot \frac{1}{2} x d x \\
& =\frac{1}{2} \int_{0}^{2} x^{2} d x=\left(\frac{x^{3}}{6}\right)_{0}^{2}=\frac{8}{6} \approx 1.333
\end{aligned}
$$

2. 

$$
\begin{aligned}
E\left(X^{2}\right) & =\int_{-\infty}^{\infty} x^{2} f(x) d x=\int_{0}^{2} x^{2} \frac{1}{2} x d x=\frac{1}{2} \int_{0}^{2} x^{3} d x=\left(\frac{x^{4}}{8}\right)_{0}^{2} \\
& =2 \\
\operatorname{Var}(X) & =E\left(X^{2}\right)-E^{2}(X)=2\left(\frac{8}{6}\right)^{2} \\
& =\frac{2}{9}
\end{aligned}
$$

## Your turn: ecology

- An ecologist wishes to mark off a circular sampling region having radius 10 m . However, the radius of the resulting region is actually a random variable $R$ with pdf:

$$
f(r)= \begin{cases}\frac{3}{2}(10-r)^{2} & 9 \leq r \leq 11 \\ 0 & \text { otherwise }\end{cases}
$$

- Calculate:

1. $E(R)$
2. $\mathrm{SD}(R)$

## Answers: ecology

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1. 

$$
\begin{aligned}
E(R) & =\int_{-\infty}^{\infty} r \cdot f(r) d r \\
& =\int_{9}^{11} r \cdot \frac{3}{2}(10-r)^{2} d r \\
& =\int_{9}^{11}\left(\frac{3}{2} r^{3}-30 r^{2}+150 r\right) d r \\
& =\left(\frac{3}{8} r^{3}-10 r^{3}+75 r^{2}\right)_{9}^{11} \\
& =\left(\frac{3}{8}(11)^{3}-10(11)^{3}+75(11)^{2}\right)-\left(\frac{3}{8} 9^{3}-10(9)^{3}+75(9)^{2}\right) \\
& =10
\end{aligned}
$$

## Answers: ecology

2. 

$$
\begin{aligned}
E\left(R^{2}\right) & =\int_{-\infty}^{\infty} r^{2} \cdot f(r) d r \\
& =\int_{9}^{11} r^{2} \cdot \frac{3}{2}(10-r)^{2} d r \\
& =\int_{9}^{11}\left(\frac{3}{2} r^{4}-30 r^{3}+150 r^{2}\right) d r \\
& =\left(\frac{3}{10} r^{5}-\frac{15}{2} r^{4}+50 r^{3}\right)_{9}^{11} \\
& =\left(\frac{3}{10}(11)^{5}-\frac{15}{2}(11)^{4}+50(11)^{3}\right)-\left(\frac{3}{10}(9)^{5}-\frac{15}{2}(9)^{4}+50(9)^{3}\right) \\
& =\frac{503}{5}=100.6 \\
\operatorname{Var}(R) & =E\left(R^{2}\right)-E^{2}(R)=\frac{503}{5}-10^{2}=\frac{3}{5}=0.6 \\
\operatorname{SD}(R) & =\sqrt{\operatorname{Var}(R)}=\sqrt{0.6} \approx 2.449
\end{aligned}
$$

## Outline

## Variance

## Functions of random variables

## Expectation of a function of a random variable

- Why does $E\left(X^{2}\right)=\int_{-\infty}^{\infty} x^{2} \cdot f(x) d x$ ?
- It turns out that for any function $g$ of a random variable:

$$
E(g(X))=\int_{-\infty}^{\infty} g(x) \cdot f(x) d x
$$

- Hence:

$$
E\left(X^{2}\right)=\int_{-\infty}^{\infty} x^{2} \cdot f(x) d x
$$

if we take $g(X)=X^{2}$.

- In the ecology example, the expected area of the circular sampling region is:

$$
E\left(\pi R^{2}\right)=\int_{-\infty}^{\infty} \pi r^{2} \cdot f(r) d r
$$

where $\pi R^{2}=g(R)$ is the sampling area.

## Expectation of a linear function of $X$

- For constants $a$ and $b$ :

$$
\begin{aligned}
E(a X+b) & =\int_{-\infty}^{\infty}(a x+b) \cdot f(x) d x \\
& =a \underbrace{\int_{-\infty}^{\infty} x \cdot f(x) d x}_{E(X)}+b \underbrace{\int_{-\infty}^{\infty} f(x) d x}_{1} \\
& =a E(X)+b
\end{aligned}
$$

Quantiles
Expected Value

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random variables

- Example: the expected diameter of the ecologist's sampling region is:

$$
E(2 \cdot R+0)=2 \cdot E(R)+0=2 \cdot 10=20
$$

## Variance of a linear function of $X$

- For constants $a$ and $b$ :

$$
\begin{aligned}
\operatorname{Var}(a X+b)= & E\left((a X+b)^{2}\right)-E^{2}(a X+b) \\
= & E\left(a^{2} X^{2}+a b X+b^{2}\right)-(a E(X)+b)^{2} \\
= & \left(a^{2} E\left(X^{2}\right)+a b E(X)+b^{2}\right) \\
& -\left(a^{2} E^{2}(X)+a b E(X)+b^{2}\right) \\
= & a^{2}\left(E\left(X^{2}\right)-E^{2}(X)\right) \\
= & a^{2} \operatorname{Var}(X)
\end{aligned}
$$

## Variance

Functions of
random variables

- Example: the variance of the diameter of the ecologist's sampling region is:

$$
\operatorname{Var}(2 \cdot R+0)=4 \operatorname{Var}(R)=4 \cdot \frac{503}{5}=\frac{2012}{5}
$$

## Standardization

- Standardization: converting a random variable $X$ into another random variable $Z$ by subtracting the mean and dividing by the standard deviation:

$$
Z=\frac{X-E(X)}{S D(X)}
$$

- $Z$ has mean 0 :

$$
\begin{aligned}
E(Z) & =E\left(\frac{X-E(X)}{S D(X)}\right)=E\left(\frac{1}{S D(X)} \cdot X-\frac{E(X)}{S D(X)}\right) \\
& =\frac{1}{S D(X)} \cdot E(X)-\frac{E(X)}{S D(X)}=0
\end{aligned}
$$

- $Z$ has variance (and standard deviation) 1:

$$
\begin{aligned}
\operatorname{Var}(Z) & =\operatorname{Var}\left(\frac{X-E(X)}{S D(X)}\right)=\operatorname{Var}\left(\frac{1}{S D(X)} \cdot X-\frac{E(X)}{S D(X)}\right) \\
& =\frac{1}{S D^{2}(X)} \operatorname{Var}(X)=\operatorname{Var}(X) \frac{1}{\operatorname{Var}(X)}=1
\end{aligned}
$$

