Continuous Random Variables: Quantiles, Expected Value, and Variance

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Quantiles of continuous distributions

The p-quantile of a random variable, X, is the number, Q(p), such that:

$$P(X \leq Q(p)) = p$$

In terms of the cumulative distribution function (cdf):

$$egin{aligned} \mathcal{F}(\mathcal{Q}(p)) &= p \ & Q(p) &= \mathcal{F}^{-1}(p) \end{aligned}$$

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Example

Let Y be the time delay (s) between a 60 Hz AC circuit and the movement of a motor on a different circuit.

$$f(y) = egin{cases} 60 & 0 < y < rac{1}{60} \ 0 & ext{otherwise} \end{cases}$$

▶ Q(0.95) :

$$0.95 = P(Y \le Q(0.95)) = \int_{-\infty}^{Q(0.95)} f(y) dy$$

= $\int_{-\infty}^{0} 0 dx + \int_{0}^{Q(0.95)} 60 dy = 0 + (60|_{0}^{Q(0.95)}$
= $60Q(0.95)$
 $Q(0.95) = \frac{0.95}{60} = \frac{19}{1200} \approx 0.0158$

Interpretation: on average, 95% of the time delays will be below 0.0158 seconds.

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Example

You can also calculate quantiles directly from the cdf:

$$F(y) = \begin{cases} 0 & y \le 0\\ 60y & 0 < y \le \frac{1}{60}\\ 1 & y > \frac{1}{60} \end{cases}$$

► Q(0.25):

$$0.25 = P(Y \le Q(0.25)) = F(Q(0.25))$$

= 60 \cdot Q(0.25)

Hence:

$$Q(0.25) = \frac{0.25}{60} = \frac{1}{240} \approx 0.00417$$

Interpretation: on average, 25% of the time delays will be below 0.00417 seconds.

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Your turn: calculating quantiles

•
$$T \sim \text{Exp}(\alpha = 1/2)$$
:

$$f(t) = egin{cases} 0 & t \leq 0 \ 2e^{-2t} & t \geq 0 \ \end{array} \quad F(t) egin{cases} 0 & t < 0 \ 1-e^{-2t} & t \geq 0 \ \end{array}$$

- Find:
 - 1. Q(0.05)
 - 2. Q(0.5)
 - 3. Q(p) for some p with $0 \le p \le 1$

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Answers: calculating quantiles

1. Q(0.05):

$$0.05 = P(T \le Q(0.05)) = F(Q(0.05)) = 1 - e^{-2Q(0.05)}$$

$$0.95 = e^{-2Q(0.05)}$$

$$og(0.95) = -2Q(0.05)$$

$$Q(0.05) = \frac{\log(0.95)}{-2} \approx 0.0256$$

2. *Q*(0.5):

I

$$\begin{array}{l} 0.5 = P(T \leq Q(0.5)) = F(Q(0.5)) = 1 - e^{-2Q(0.5)} \\ 0.5 = e^{-2Q(0.5)} \\ \log(0.5) = -2Q(0.5) \\ Q(0.5) = \frac{\log(0.5)}{-2} \approx 0.347 \end{array}$$

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Answers: calculating quantiles

3. Q(p)

$$p = P(T \le Q(p)) = F(Q(p)) = 1 - e^{-2Q(p)}$$
$$1 - p = e^{-2Q(p)}$$
$$\log(1 - p) = -2Q(p)$$
$$Q(p) = \frac{\log(1 - p)}{-2}$$

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The expected value of a continuous random variable is:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

► As with continuous random variables, E(X) (often denoted by µ) is the mean of X, a measure of center. Continuous Random Variables: Quantiles, Expected Value, and Variance

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Example: time delay, Y

$$f(y) = egin{cases} 60 & 0 \leq y \leq rac{1}{60} \ 0 & ext{otherwise} \end{cases}$$

$$E(Y) = \int_{-\infty}^{\infty} y \cdot f(y) dy$$

= $\int_{-\infty}^{0} y \cdot 0 dy + \int_{0}^{1/60} y \cdot 60 dy + \int_{1/60}^{\infty} y \cdot 0 dy$
= $0 + \left(\frac{y^2}{2} \cdot 60\right)_{0}^{1/60} + 0$
= $\frac{1}{2} \left(\frac{1}{60}\right)^2 \cdot 60 = \frac{1}{120}$

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E(X) is the "center of mass" of a distribution



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Your turn: calculate E(X)

$$f(x) = egin{cases} 0 & x < 0 \ rac{1}{lpha} e^{-x/lpha} & x \ge 0 \end{cases}$$

1. $X \sim \text{Exp}(3)$ 2. $X \sim \text{Exp}(\alpha)$ Continuous Random Variables: Quantiles, Expected Value, and Variance

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Answers: Calculate E(X)

1. $X \sim Exp(3)$:

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx$$
$$= \int_{-\infty}^{0} x \cdot 0 dx + \int_{0}^{\infty} x \cdot \frac{1}{3} e^{-x/3} dx$$

integration by parts:

$$= 0 + \left(x(-e^{-x/3})\right)_0^\infty - \int_0^\infty (-e^{-x/3})dx$$

= $\left(-\infty e^{-\infty/3} + 0e^{-0/3}\right) + \int_0^\infty e^{-x/3}dx$
= $0 + \left(-3e^{-x/3}\right)_0^\infty$
= $\left(-3e^{-\infty/3} + 3e^{-0/3}\right)$
= 3

2. Similarly, $E(X) = \alpha$ when $X \sim Exp(\alpha)$.

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Example: waiting time for the next student to arrive at the library

- From 12:00 to 12:10 PM, about 12.5 students per minute enter on average.
- Hence, the average waiting time for the next student is $\frac{1}{12.5} = 0.08$ minutes for the next student.
- Let $T \sim \text{Exp}(0.08)$ be the time until the next student arrives.
- P(wait is more than 10 seconds) =

$$P(T > 1/6) = 1 - F(1/6) = 1 - (1 - e^{(-0.08 \cdot 1/6)}) = 0.12$$



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• The variance of a continuous random variable X is:

$$\operatorname{Var}(X) = \int_{-\infty}^{\infty} (x - E(X))^2 \cdot f(x) dx$$

Shortcut formulas:

$$Var(X) = \int_{-\infty}^{\infty} x^2 f(x) dx - E^2(X)$$
$$= E(X^2) - E^2(X)$$

• The standard deviation is $SD(X) = \sqrt{Var(X)}$

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Your turn: checkout time

Let X denote the amount of time for which a book on 2-hour reserve at a college library is checked out by a randomly selected student and suppose that X has density function

$$f(x) = \begin{cases} .5x & 0 \le x \le 2\\ 0 & \text{otherwise} \end{cases}$$

Calculate:

E(X)
 Var(X)

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Answers: checkout time

1.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx = \int_{0}^{2} x \cdot \frac{1}{2} x dx$$
$$= \frac{1}{2} \int_{0}^{2} x^{2} dx = \left(\frac{x^{3}}{6}\right)_{0}^{2} = \frac{8}{6} \approx 1.333$$

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$$E(X^{2}) = \int_{-\infty}^{\infty} x^{2} f(x) dx = \int_{0}^{2} x^{2} \frac{1}{2} x dx = \frac{1}{2} \int_{0}^{2} x^{3} dx = \left(\frac{x^{4}}{8}\right)_{0}^{2}$$

= 2
$$Var(X) = E(X^{2}) - E^{2}(X) = 2\left(\frac{8}{6}\right)^{2}$$

= $\frac{2}{9}$

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Your turn: ecology

An ecologist wishes to mark off a circular sampling region having radius 10 m. However, the radius of the resulting region is actually a random variable R with pdf:

$$f(r) = egin{cases} rac{3}{2}(10-r)^2 & 9 \leq r \leq 11 \ 0 & ext{otherwise} \end{cases}$$

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- Calculate:
 - 1. E(R)
 - 2. SD(*R*)

Answers: ecology

1.

$$\begin{split} E(R) &= \int_{-\infty}^{\infty} r \cdot f(r) dr \\ &= \int_{9}^{11} r \cdot \frac{3}{2} (10 - r)^2 dr \\ &= \int_{9}^{11} \left(\frac{3}{2}r^3 - 30r^2 + 150r\right) dr \\ &= \left(\frac{3}{8}r^3 - 10r^3 + 75r^2\right)_{9}^{11} \\ &= \left(\frac{3}{8}(11)^3 - 10(11)^3 + 75(11)^2\right) - \left(\frac{3}{8}9^3 - 10(9)^3 + 75(9)^2\right) \\ &= 10 \end{split}$$

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Answers: ecology

2.

$$\begin{split} E(R^2) &= \int_{-\infty}^{\infty} r^2 \cdot f(r) dr \\ &= \int_{9}^{11} r^2 \cdot \frac{3}{2} (10 - r)^2 dr \\ &= \int_{9}^{11} \left(\frac{3}{2} r^4 - 30r^3 + 150r^2 \right) dr \\ &= \left(\frac{3}{10} r^5 - \frac{15}{2} r^4 + 50r^3 \right)_{9}^{11} \\ &= \left(\frac{3}{10} (11)^5 - \frac{15}{2} (11)^4 + 50(11)^3 \right) - \left(\frac{3}{10} (9)^5 - \frac{15}{2} (9)^4 + 50(9)^3 \right) \\ &= \frac{503}{5} = 100.6 \\ \operatorname{Var}(R) &= E(R^2) - E^2(R) = \frac{503}{5} - 10^2 = \frac{3}{5} = 0.6 \\ \operatorname{SD}(R) &= \sqrt{\operatorname{Var}(R)} = \sqrt{0.6} \approx 2.449 \end{split}$$

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Expectation of a function of a random variable

• Why does
$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x) dx$$
?

It turns out that for any function g of a random variable:

$$E(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f(x) dx$$

Hence:

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x) dx$$

if we take $g(X) = X^2$.

In the ecology example, the expected area of the circular sampling region is:

$$E(\pi R^2) = \int_{-\infty}^{\infty} \pi r^2 \cdot f(r) dr$$

where $\pi R^2 = g(R)$ is the sampling area.

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Expectation of a linear function of X

For constants *a* and *b*:

$$E(aX + b) = \int_{-\infty}^{\infty} (ax + b) \cdot f(x) dx$$

= $a \underbrace{\int_{-\infty}^{\infty} x \cdot f(x) dx}_{E(X)} + b \underbrace{\int_{-\infty}^{\infty} f(x) dx}_{1}$
= $\boxed{aE(X) + b}$

Example: the expected *diameter* of the ecologist's sampling region is:

$$E(2 \cdot R + 0) = 2 \cdot E(R) + 0 = 2 \cdot 10 = 20$$

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Variance of a linear function of X

For constants *a* and *b*:

$$Var(aX + b) = E((aX + b)^{2}) - E^{2}(aX + b)$$

= $E(a^{2}X^{2} + abX + b^{2}) - (aE(X) + b)^{2}$
= $(a^{2}E(X^{2}) + abE(X) + b^{2})$
 $- (a^{2}E^{2}(X) + abE(X) + b^{2})$
= $a^{2}(E(X^{2}) - E^{2}(X))$
= $\boxed{a^{2}Var(X)}$

Example: the variance of the *diameter* of the ecologist's sampling region is:

$$Var(2 \cdot R + 0) = 4Var(R) = 4 \cdot \frac{503}{5} = \frac{2012}{5}$$

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Standardization

Standardization: converting a random variable X into another random variable Z by subtracting the mean and dividing by the standard deviation:

$$Z = \frac{X - E(X)}{SD(X)}$$

Z has mean 0:

$$E(Z) = E\left(\frac{X - E(X)}{SD(X)}\right) = E\left(\frac{1}{SD(X)} \cdot X - \frac{E(X)}{SD(X)}\right)$$
$$= \frac{1}{SD(X)} \cdot E(X) - \frac{E(X)}{SD(X)} = 0$$

Z has variance (and standard deviation) 1:

$$\begin{aligned} \mathsf{Var}(Z) &= \mathsf{Var}\left(\frac{X - E(X)}{SD(X)}\right) = \mathsf{Var}\left(\frac{1}{SD(X)} \cdot X - \frac{E(X)}{SD(X)}\right) \\ &= \frac{1}{SD^2(X)}\mathsf{Var}(X) = \mathsf{Var}(X)\frac{1}{\mathsf{Var}(X)} = 1 \end{aligned}$$

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